OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4754(B)

Applications of Advanced Mathematics (C4)

Paper B: Comprehension

INSERT

Monday **23 JANUARY 2006** Afternoon Up to 1 hour

INSTRUCTIONS TO CANDIDATES

- This insert contains the text for use with the questions.

This insert consists of 8 printed pages.

Electing Members of the European Parliament

The Regional List System

The British members of the European Parliament are elected using a form of proportional representation called the Regional List System. This article compares two different ways of working out who should be elected.

Great Britain is divided into 11 regions and each of these is assigned a number of seats in the European Parliament. So, for example, the South West region has 7 seats, meaning that it elects 7 members to the parliament.

Each political party in a region presents a list of candidates in order of preference. For example, in a region with 5 seats, Party A could present a list like that in Table 1.

	Party A
1	Comfort Owosu
2	Graham Reid
3	Simon White
4	Malini Ghosh
5	Sam Roy

Table 1

According to the proportion of the votes that Party A receives, 0, 1, 2, 3, 4 or all 5 of the people on the list may be elected.

Imagine an election for 6 seats in one region. It is contested by 8 political parties, A, B, C, D, E, F, G and H and the percentages of votes they receive are given in Table 2.

Party	Votes (%)
A	22.2
B	6.1
C	27.0
D	16.6
E	11.2
F	3.7
G	10.6
H	2.6

Table 2

How do you decide which parties get the 6 seats?

The Trial-and-Improvement method

The Regional List System is based on the idea that, in any particular regional election, a certain percentage of votes will win one seat. In this article, this is called the *acceptance percentage* and is denoted by $a\%$. A party which receives less than $a\%$ of the votes is given no seats; one that receives at least $a\%$ and less than $2a\%$ of the votes gets 1 seat; at least $2a\%$ and less than $3a\%$ of the votes translates into 2 seats and so on.

At first sight it might seem that, in the example in Table 2, since $100\% \div 6 = 16\frac{2}{3}\%$, the acceptance percentage should be about 16.7% of the votes. Clearly that will not work since it would give Parties A and C one seat each and none of the others would get any. Only 2 members would be elected rather than the required 6.

So what percentage of the votes is needed for exactly 6 people to be elected? One method of deciding is to try out different possible acceptance percentages and find one which results in 6 seats. In Table 3, values of a of 8, 10, 12 and 14 are tried out.

Acceptance percentage, $a\%$		**8%**	**10%**	**12%**	**14%**
Party	**Votes (%)**	**Seats**	**Seats**	**Seats**	**Seats**
A	22.2	2	2	1	1
B	6.1	0	0	0	0
C	27.0	3	2	2	1
D	16.6	2	1	1	1
E	11.2	1	1	0	0
F	3.7	0	0	0	0
G	10.6	1	1	0	0
H	2.6	0	0	0	0
Total seats		**9**	**7**	**4**	**3**

Table 3

Table 3 shows that an acceptance percentage of 10% is too low for 6 seats and one of 12% is too high. So it is natural to try 11%. This is shown in Table 4.

Party	Votes (%)	$a\% = 11\%$ Seats	Used Votes (%)	Unused Votes (%)
A	22.2	2	22	0.2
B	6.1	0	0	6.1
C	27.0	2	22	5.0
D	16.6	1	11	5.6
E	11.2	1	11	0.2
F	3.7	0	0	3.7
G	10.6	0	0	10.6
H	2.6	0	0	2.6
Total		**6**	**66.0%**	**34.0%**

Table 4

Table 4 shows that an acceptance percentage of 11% gives 2 seats each to Parties A and C and one each to D and E, making a total of 6 in all. Party G just misses out. Thus with these voting figures, and with 6 seats to be allocated, 11% is a suitable acceptance percentage.

This Trial-and-Improvement method involves finding an interval within which the acceptance percentage must lie, in this example between 10% and 12%, and then closing in on a suitable value. It is like solving an equation by a change of sign method, but with the difference that in this case there is a range of possible answers: in the example above any value greater than 10.6% up to and including 11.1% will give a satisfactory acceptance percentage.

The range of values that an acceptance percentage can take depends on the number of seats.

The d'Hondt Formula

A different method of allocation is provided by the d'Hondt Formula. This is illustrated in Table 5, using the same data as before.

	Round						
Party	**1**	**2**	**3**	**4**	**5**	**6**	**Residual**
A	22.2	22.2	11.1	11.1	11.1	11.1	7.4
B	6.1	6.1	6.1	6.1	6.1	6.1	6.1
C	27.0	13.5	13.5	13.5	9.0	9.0	9.0
D	16.6	16.6	16.6	8.3	8.3	8.3	8.3
E	11.2	11.2	11.2	11.2	11.2	5.6	5.6
F	3.7	3.7	3.7	3.7	3.7	3.7	3.7
G	10.6	10.6	10.6	10.6	10.6	10.6	10.6
H	2.6	2.6	2.6	2.6	2.6	2.6	2.6
Seat allocated to	**C**	**A**	**D**	**C**	**E**	**A**	

Table 5

The d'Hondt Formula provides an algorithm in which seats are allocated from the top down. Each time a party is allocated a seat its vote is replaced by $\frac{V}{n+1}$, where V is the percentage of votes it received originally and n is the number of seats it has now been allocated.

- In Round 1, Party C has the highest percentage, 27.0, so C gets the first seat.
- For Round 2, the vote for Party C is divided by $(n + 1)$, with n taking the value 1 since the party has now been allocated 1 seat. So the figure 27.0 for C is replaced by $27.0 \div 2 = 13.5$.
- The highest figure in Round 2 is 22.2 for Party A and so the next seat goes to A. The figure 22.2 for A is replaced by $22.2 \div 2 = 11.1$ in Round 3. The seat for Round 3 is allocated to Party D.
- In Round 4, Party C gets a second seat so that the value of n for this party is now 2. So the original figure for C is now divided by $(2 + 1) = 3$ for Round 5; $27.0 \div 3 = 9.0$.
- The figures in the final column, headed "Residual", are those that would be used if an extra seat were to be allocated. They do not have the same meaning as "Unused Votes" in Table 4.

In this example, the outcome obtained using the d'Hondt Formula is the same as that from the Trial-and-Improvement method, namely 2 seats each for A and C, and one each for D and E. Again Party G just misses out; if there had been 7 seats G would have got the last one.

Equivalence of the two methods

Since the two methods are completely different, it comes as something of a surprise that in this example they produce the same outcome. The question then arises as to whether they will always produce the same outcome.

The results of the real elections are worked out using the d'Hondt Formula; if there were circumstances in which this produced different outcomes from the Trial-and-Improvement method, there might be doubt about the fairness of the election.

It is, however, possible to show that the outcomes from the two methods will always be the same.

Before seeing how to do this, it is important to understand that there are fundamental differences between the methods.

- The Trial-and-Improvement method is based on finding an acceptance percentage for the particular number of seats; for a different number of seats you have to find a different acceptance percentage.
- Using the d'Hondt Formula, an acceptance percentage is never known. The method gives the outcome round by round for as many seats as are to be allocated.

In the Trial-and-Improvement method, call the parties Party 1, Party 2, … , Party m.

Suppose that Party k receives V_k% of the votes and is allocated N_k seats.

One way of looking at this outcome is that each of the N_k elected members of parliament received an acceptance percentage of the votes, $a\%$, and then there were some "unused votes" left over, as shown in Table 4.

The percentage of unused votes must be less than the acceptance percentage; otherwise the party would have been allocated another seat.

Therefore $$V_k - (N_k \times a) < a$$

and so $$a > \frac{V_k}{N_k + 1}.$$

This is true for all the values of k from 1 to m.

A second condition arises for those parties that have been allocated seats. If such a party had been allocated one fewer seat, $N_k - 1$ instead of N_k, the percentage of votes left over would have been at least the acceptance percentage.

Therefore $$V_k - (N_k - 1)\,a \geqslant a$$

and so $$a \leqslant \frac{V_k}{N_k}.$$

(If, however, a party has not been allocated any seats anyway, then there is no equivalent second inequality.)

Thus $$\frac{V_k}{N_k + 1} < a \leqslant \frac{V_k}{N_k} \qquad \text{if } N_k > 0$$

and $$\frac{V_k}{N_k + 1} < a \qquad \text{if } N_k = 0.$$

Now look at Table 6 below. This reproduces row C from Table 5 illustrating the d'Hondt Formula.

	Round						
Party	**1**	**2**	**3**	**4**	**5**	**6**	**Residual**
C	27.0	13.5	13.5	13.5	9.0	9.0	9.0
Seat allocated to	**C**			**C**			

Table 6

Using the notation above, with C as Party 3, this becomes Table 7.

	Round						
Party	**1**	**2**	**3**	**4**	**5**	**6**	**Residual**
C	V_3	$\frac{1}{2}V_3$	$\frac{1}{2}V_3$	$\frac{1}{2}V_3$	$\frac{1}{3}V_3$	$\frac{1}{3}V_3$	$\frac{1}{3}V_3$
Seat allocated to	**C**			**C**			

Table 7

So in this case the acceptance percentage lies between $\frac{1}{2}V_3$, which gives another seat in Round 4, and $\frac{1}{3}V_3$ which does not give another seat.

So $$\tfrac{1}{3}V_3 < a \leqslant \tfrac{1}{2}V_3.$$

This result can be generalised by replacing Party 3 by Party k, and the 2 seats by N_k seats, to obtain the result found above for the Trial-and-Improvement method,

$$\frac{V_k}{N_k + 1} < a \leqslant \frac{V_k}{N_k}.$$

Thus the two methods are indeed equivalent.

Discovering the d'Hondt Formula

While the Trial-and-Improvement method is straightforward, the d'Hondt Formula is quite subtle, so much so that it is natural to ask "How did anyone think this up in the first place?"

The method owes its name to Victor d'Hondt, a Belgian lawyer and mathematician who first described it in 1878. It is used in many countries, including the United States where it is called the Jefferson Method.

The graph in Fig. 8 provides a clue as to how it might have been discovered. Fig. 8 shows the ranges of possible acceptance percentages for different numbers of seats, for the figures in Table 2. To draw such a graph, you need to work out the end-points of the various ranges. The range $10.6 < a \leqslant 11.1$ for the case of electing 6 people was given on line 37. These end-points turn out to be the largest numbers in successive Rounds in Table 5 which illustrates the d'Hondt Formula. Thus drawing this type of graph leads you into the d'Hondt Formula.

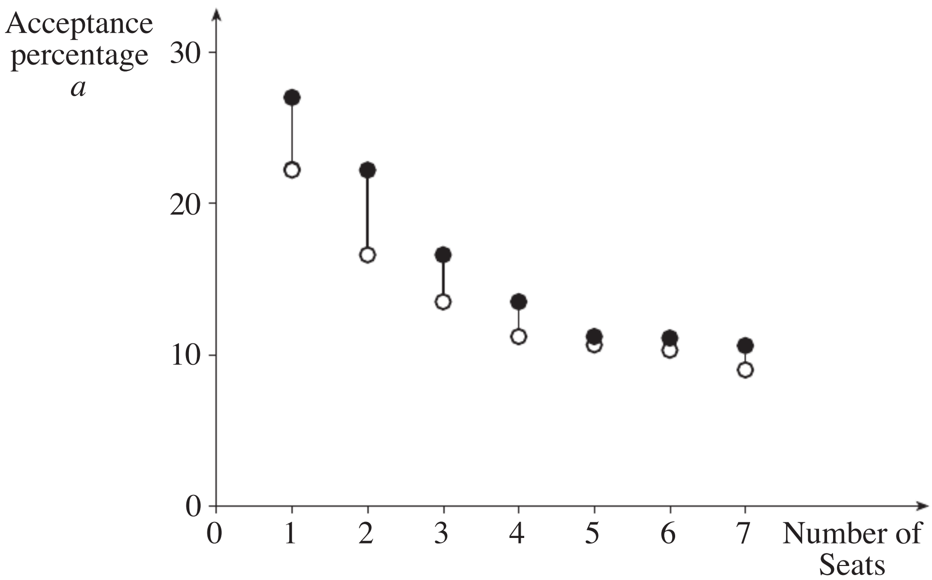

Fig. 8

[Turn over

Which is the better method?

Since the two methods are equivalent, there is no mathematical reason to declare either to be the better.

There are, however, two other considerations.

- Is one method easier than the other to apply?
- Is one method easier than the other for the public to understand, and so more likely to generate confidence in the outcome?